Non-Archimedean Integration and Strict Topologies

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Introduction

Let $C_b(X, E)$ be the space of all bounded continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. By $C_{rc}(X, E)$ we denote the space of all $f \in C_b(X, E)$ for which $f(X)$ is a relatively compact subset of E. In section 2 of this paper we show that, if E is polar and complete and Y a closed subset of X which is either compact or X is ultranormal, then there exists a linear map $T : C_{rc}(Y, E) \to C_{rc}(X, E)$ such that Tf is an extension of f and $\|Tf\|_p = \|f\|_p$ for all $f \in C_{rc}(Y, E)$ and every polar continuous seminorm p on E. Using this we identify in section 3 the completion of the space $C_b(X, E)$ under the strict topology β_o when E is polar. If $K(X)$ is the algebra of all clopen (i.e. both closed and open) subsets of X, we define in section 4 the product of certain $\mathbb{K}$-valued finitely-additive measures on $K(X)$ with E'-valued measures on $K(Y)$, where Y is another zero-dimensional topological space. Finally in sections 5 and 6 we define the so called (VR)-integral and Q-integral of functions in E^X with respect to certain measures on $K(X)$.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\mathbb{K}$, we mean a non-Archimedean seminorm. Similarly, by a locally convex space we mean a non-Archimedean locally convex space over $\mathbb{K}$. For E a locally convex space, we denote by $cs(E)$ the collection of all continuous seminorms on E, by E' its dual space and by $\hat{E}$ its completion. If F is another locally convex space, then $E \otimes F$ will be the tensor product of E, F with the projective topology.

Let now X be a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [4]) and by $\upsilon_o X$ the $\mathbb{N}$-repletion of X ($\mathbb{N}$ is the set of natural numbers), i.e. the subspace of $\beta_o X$ consisting of all $x \in \beta_o X$ with the following property: For each sequence (V_n) of neighborhoods of x in $\beta_o X$ we have that $\bigcap V_n \cap X \neq \emptyset$. The space X is called $\mathbb{N}$-replete if $X = \upsilon_o X$. We will denote by $C_b(X,E)$ the space of all bounded continuous E-valued functions on X and by $C_{rc}(X,E)$ the space of all $f \in C_b(X,E)$ for which $f(X)$ is relatively compact in E. In case $E = \mathbb{K}$, we will simply write $C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A in X and by $\overline{A}^{\beta_o X}$ the closure of A in $\beta_o X$. Every $f \in C_{rc}(X,E)$ has a unique continuous extension f^{β_o} to all of $\beta_o X$. For f an E-valued function on X, p a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X,E)$ (see [7]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for each $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if x is not in Y. Let Ω be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ whose continuous extension h^{β_o} vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X,E)$ generated by the seminorms $f \mapsto \|hf\|_p, h \in C_H$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω, is denoted by β_p while β is the projective limit of the topologies $\beta_p, p \in cs(E)$. The following Theorem is proved in [11].

Theorem 1.1 *An absolutely convex subset V of $C_b(X,E)$ is a $\beta_{H,p}$-neighborhood of zero iff the following condition is satisfied: For each $r > 0$, there exist $\epsilon > 0$ and a clopen subset A of X, with $\bar{A}^{\beta_o X} \cap H = \emptyset$, such that*

$$\{f \in C_b(X,E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset V.$$

Let now $K(X)$ be the algebra of all clopen, (i.e. closed and open) subsets of X. We denote by $M(X,E')$ (see [6]) the space of all finitely-additive E'-valued measures m on $K(X)$ for which $m(K(X))$ is an equicontinuous subset of E'. For each m in $M(X,E')$ there exists $p \in cs(E)$ with $m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, A \supset B \in K(X)\}.$$

The space of all $m \in M(X,E')$ with $m_p(X) < \infty$ is denoted by $M_p(X,E')$. We denote by $M_\tau(X,E')$ the space of all $m \in M(X,E')$ such that, for every decreasing net (A_δ) of clopen subsets of X, with $\cap A_\delta = \emptyset$, there exists $p \in cs(E)$ such that $m_p(A_\delta) \to 0$. Also by $\mathcal{M}_{\tau,p}(X,E')$ we denote the space of all $m \in M_p(X,E')$ such that $m_p(A_\delta) \to 0$ for every decreasing net (A_δ) of clopen subsets of X with $\cap A_\delta = \emptyset$. Let

$$\mathcal{M}_\tau(X,E') = \bigcup_{p \in cs(E)} \mathcal{M}_{\tau,p}(X,E').$$

For $p \in cs(E)$, we denote by $M_{t,p}(X,E')$ the space of all $m \in M_p(X,E')$ for which m_p is tight, i.e. for every $\epsilon > 0$, there exists a compact subset Y of X such that

$m_p(A) \leq \epsilon$ if A is disjoint from Y. We define

$$M_t(X, E') = \bigcup_{p \in cs(E)} M_{t,p}(X, E').$$

As it is shown in [11], $\mathcal{M}_{\tau,p}(X, E') = M_{t,p}(X, E')$. In case $E = \mathbb{K}$, we write $M(X), M_\tau(X)$ and $M_t(X)$ for $M(X, E'), M_\tau(X, E')$ and $M_t(X, E')$, respectively. Also, for $\mu \in M(X)$, we define $|\mu|(A) = \mu_p(A)$, where $p = |.|$ is the valuation of $\mathbb{K}$.

Next, we recall the definition of the integral of an E-valued function f on X with respect to an $m \in M(X, E')$. For $A \in K(X), A \neq \emptyset$, let $\mathcal{D}_A$ denote the family of all $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_i \in A_i$. We make $\mathcal{D}_A$ a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For $f \in E^X, m \in M(X, E')$ and $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, we define $\omega_\alpha(f, m) = \sum_{i=1}^n m(A_i) f(x_i)$. If the $\lim_\alpha \omega_\alpha(f, m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in K(X)$ and $\int_A f dm = \int \chi_A f dm$. Every $m \in M(X, E')$ defines a τ_u-continuous linear functional on $C_{rc}(X, E)$ by $f \mapsto \int f dm$ (see [6]). Also every $\phi \in (C_{rc}(X, E), \tau_u)'$ is given in this way by a unique m.

As it is shown in [7], every $m \in M_t(X, E')$ defines a β_o-continuous linear form on $C_b(X, E)$ by $u_m(f) = \int f dm$. Moreover the map $m \mapsto u_m$, from $M_t(X, E')$ to $(C_b(X, E), \beta_o)'$, is an algebraic isomorphism. Also it is shown in [11] that every $f \in C_b(X, E)$ is m-integrable, for every $m \in M_\tau(X, E'$, and the map u_m is β-continuous. Moreover, every element of $(C_b(X, E), \beta)'$ is given in this way for a unique $m \in M_\tau(X, E')$. For all unexplained terms on locally convex spaces we refer to [15] and [16].

Throughout the paper, unless it is stated explicitly othewise, X is a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space.

2 Extensions of Continuous Functions

The classical Tietze's extension Theorem states that, for a Hausdorff tropological space X, the following are equivalent: 1) X is normal.
2) For every closed subset Y of X and each continuous function $f : Y \to \mathbb{R}$, which is bounded (equivalently for which $f(X)$ is relatively compact), there exists a continuous extension $\bar{f} : X \to \mathbb{R}$ such that $\sup\{|f(x)| : x \in Y\} = \sup\{|\bar{f}(x)| : x \in X\}$
In this section we will examine the extension problem when we replace $\mathbb{R}$ by a complete non-Archimedean locally convex space E.

Lemma 2.1 *Let E be a Hausdorff locally convex space, $E \neq \{0\}$. If X is a Hausdorff topological space such that, for any closed subset Y of X and any $f \in C_{rc}(Y, E)$, there exists a continuous extension $\bar{f} : X \to E$ of f, then X is ultranormal.*

Proof: Let A, B be disjoint closed subsets of X and let a be a nonzero element of E. The function $f : A \cup B \to E, f(x) = 0$ if $x \in A$ and $f(x) = a$ if $x \in B$ is continuous. If g is a continuous extension of f and V a clopen neighborhood of zero in E not containing a, then $g^{-1}(V)$ is a a clopen subset of X containing A and disjoint from B, which proves that X is ultranormal.

Assume now that Y is a closed subset of X and that either Y is compact or X is ultranormal. In both cases, for every clopen in Y subset A of Y there exists a clopen subset B of X with $A = B \cap Y$. By [16], Corollary 5.23, there exists a family $(A_i)_{i\in I}$ of clopen in Y subsets of Y such that the family $\{\chi_{A_i} : i \in I\}$ of the corresponding characteritic functions is an orthonormal basis in $C_{rc}(Y)$ for the tropology of uniform convergence on $C_{rc}(Y)$. For each $i \in I$, choose a clopen subset $\tilde{A}_i$ of X whose intersection with Y is A_i. Then, as it is shown in the proof of Theorem 5.24 in [16], there exists a linear isometry $S : C_{rc}(Y) \to C_{rc}(X)$ such that $f(\chi_{A_i}) = \chi_{\tilde{A}_i}$ and Sg is an extension of g for every $g \in C_{rc}(Y)$.

Theorem 2.2 *Let $X, Y, (A_i)_{i\in I}$ and S be as above and assume that E is polar and complete. Then, there exists a linear map*

$$T : C_{rc}(Y, E) \to C_{rc}(X, E)$$

such that Tf is an extension of f and $\|Tf\|_p = \|f\|_p$ for all $f \in C_{rc}(Y, E)$ and all polar $p \in cs(E)$. Moreover, if $f = \sum_{i\in I} \chi_{A_i} s_i$, then $Tf = \sum_{i\in I} \chi_{\tilde{A}_i} s_i$ where convergence of the sums is with respect to the corresponding topologies of uniform convergence.

Proof: Claim I: If J is a finite subset of I, $f = \sum_{i\in J} \chi_{A_i} s_i, h = \sum_{i\in I} \chi_{\tilde{A}_i} s_i, s_i \in E$, then $\|h\|_p \le \|f\|_p$ (and hence $\|h\|_p = \|f\|_p$)
Indeed, given $\epsilon > 0$, there exists $x \in X$ with $p(h(x)) > \|h\|_p - \epsilon$. As p is polar, there exists $\phi \in E', |\phi| \le p$, such that $|\phi(h(x))| > \|h\|_p - \epsilon$. Since $S\left(\sum_{i\in J} \phi(s_i)\chi_{A_i}\right) = \sum_{i\in J} \phi(s_i)\chi_{\tilde{A}_i}$, we have that

$$\|f\|_p \ge \|\sum_{i\in J} \phi(s_i)\chi_{A_i}\| = \|\sum_{i\in J} \phi(s_i)\chi_{\tilde{A}_i}\| \ge |\phi(h(x)| > \|h\|_p - \epsilon,$$

and the claim follows.
Claim II : If G is the subspace of all $f \in C_{rc}(Y, E)$ which can be written in the form $f = \sum_{i\in J} \chi_{A_i} s_i$, where all but a finite number of the s_i are zero, then G is τ_u-dense in $C_{rc}(Y, E)$.
To show this, we first observe that every $f \in G$ can be written uniquely in the form $f = \sum_{i\in J} \chi_{A_i} s_i$. In fact assume that $f = \sum_{i\in J_1} \chi_{A_i} s_i = \sum_{i\in J_2} \chi_{A_i} u_i$, where J_1, J_2 are finite subsets of I. We may assume that $J_1 = J_2 = J$. For each $\phi \in E'$, we have that $\sum_{i\in J} \phi(s_i)\chi_{A_i} = \sum_{i\in J} \phi(u_i)\chi_{\tilde{A}_i}$ and so $\phi(s_i) = \phi(u_i)$, for all $i \in J$, which implies that $s_i = u_i$ since E is Hausdorff and polar. Let now $f \in C_{rc}(Y, E)$ and a polar $p \in cs(E)$. There exist a finite clopen partition $\{D_1, \dots, D_n\}$ of Y and $x_i \in D_i$ such that $\|f - \sum_{k=1}^n \chi_{D_k} f(x_k)\|_p < 1$. Let A be a clopen subset of Y. Then $\chi_A = \sum_{i\in I} \alpha_i \chi_{A_i}, \alpha_i \in \mathbb{K}$, and so $\chi_A s = \sum_{i\in I} \alpha_i \chi_{A_i} s$ for all $s \in E$. To finish the proof of our claim, it suffices to prove that every $\chi_A s$ is in the closure of G in $C_{rc}(Y, E)$. So let q be a polar continuous seminorm on E and $\epsilon > 0$. There exists a finite subset J of I such that $\|\chi_A s - \sum_{i\in J} \alpha_i \chi_{A_i} s\|_q = q(s)\|\chi_A - \sum_{i\in J} \alpha_i \chi_{A_i}\|_q < \epsilon$, which proves that $\chi_A s \in \bar{G}$. This completes the proof of our claim.

Claim III: There exists a continuous linear map $T : C_{rc}(Y, E) \to C_{rc}(X, E)$ such that $T(f) = \sum_{i\in I} \chi_{\bar{A}_i} s_i$ for $f = \sum_{i\in I} \chi_{A_i} s_i$ in G. Indeed, define

$$T : G \to C_{rc}(X, E), \quad T(\sum_{i\in I} \chi_{A_i} s_i) = \sum_{i\in I} \chi_{\bar{A}_i} s_i.$$

Then T is well defined and linear. Moreover $\|Tf\|_p = \|f\|_p$ for each $f \in G$ and each polar $p \in cs(E)$. Since E is complete, the space $C_{rc}(X, E)$, with the topology of uniform convergence τ_u, is complete and hence (by Claim II) there exists a unique continuous extension of T to all of $C_{rc}(Y, E)$. We denote also by T this extension. If p is a polar continuous seminorm on E and $f \in C_{rc}(Y, E)$, then there exists a net (f_δ) in G converginmg to f. Thus $\|Tf\|_p = \lim \|Tf_\delta\|_p = \lim \|f_\delta\|_p = \|f\|_p$. Since Tf_δ is an extension of f_δ, it follows that Tf is an extension of f. This completes the proof.

For $p \in cs(E)$, let $M_{k,p}(X, E')$ be the space of all $m \in M_p(X, E')$ which have a compact support, i.e. there exists a compact subset Y of X such that $m(A) = 0$ if A is disjoint from Y. Let $m \in M_p(X, E')$, where $p \in cs(E)$. We will denote also by p the unique continuous extension of p to all of $\hat{E}$. If $\phi \in E'$ is such that $|\phi| \le p$, then there exists a unique continuous extension $\hat{\phi}$ of ϕ to all of $\hat{E}$. For each $A \in K(X)$, let $\hat{m}(A)$ be the continuous extension of $m(A)$. Then $\hat{m} \in M_p(X, \hat{E}')$ and $\hat{m}_p(A) = m_p(A)$. In fact, it is clear that $m_p(A) \le \hat{m}_p(A)$. On the other hand, let B be contained in A and let $s \in \hat{E}, s \neq 0$. If $\hat{m}(B)s \neq 0$, then there exists $u \in E$ with $p(s - u) < p(s)$ and $|\hat{m}(B)(s - u)| < |\hat{m}(B)s|$ Now $p(s) = p(u)$ and $|\hat{m}(B)s| = |\hat{m}(B)u|$. It follows easily from this that $\hat{m}_p(A) \le m_p(A)$, and the claim follows. It is also clear that $\hat{m} \in M_{t,p}(X, \hat{E}')$ if $m \in M_{t,p}(X, E')$.

As an application of the preceding Theorem , we get the following

Theorem 2.3 *Assume that E is polar and let p be a polar continuous seminorm on E. If we consider on $M_p(X, E')$ the norm $\|m\|_p = m_p(X)$, then $M_{t,p}(X, E')$ coincides with the closure of $M_{k,p}(X, E')$ in $M_p(X, E')$.*

Proof: Let $m \in M_p(X, E')$ be in the closure of $M_{k,p}(X, E')$. Given $\epsilon > 0$, choose $\bar{m} \in M_{k,p}(X, E')$ such that $\|m - \bar{m}\|_p < \epsilon$. Let Y be a compact support for $\bar{m}$. If $A \in K(X)$ is disjoint from Y, then for $B \subset A$ and $s \in E$ we have $|m(B)s| = |[m(B) - \bar{m}(B)]s| \le \|m - \bar{m}\|_p p(s)$ and so $m_p(A) \le \epsilon$, which proves that $m \in M_{t,p}(X, E')$. Conversely, let $m \in M_{t,p}(X, E')$. Then $\hat{m} \in M_{t,p}(X, \hat{E}')$. Let Y be a compact subset of X such that $m_p(A) = \hat{m}_p(A) \le \epsilon$ if A is disjoint from Y. Since $\hat{E}$ is complete and polar, there exists a linear map $S : C_{rc}(Y, \hat{E}) \to C_{rc}(X, \hat{E})$ such that, for each $f \in C_{rc}(Y, \hat{E}), Sf$ is an extension of f and $\|Sf\|_q = \|f\|_q$ for each continuous polar seminorm q on $\hat{E}$. Define

$$\phi : C_{rc}(X, \hat{E}) \to \mathbb{K}, \phi(f) = \int S(f|Y) d\hat{m}.$$

Then

$$|\phi(f)| \le m_p(X) \|S(f|Y)\|_p = m_p(X) \|f\|_{Y,p} \le m_p(X) \|f\|_p.$$

Hence, there exists $\mu \in M_p(X, \hat{E}')$ such that $\phi(f) = \int f d\mu$ for all $f \in C_{rc}(Y, \hat{E})$. Then Y is a support set for μ. Let $\bar{m} : K(X) \to E', \bar{m}(A) = \mu(A)|E$. Then $\bar{m} \in M_{k,p}(X, E')$. Finally, if $|\lambda| > 1$, then $\|\bar{m} - m\| \leq \epsilon|\lambda|$. Indeed, let $s \in E$ with $p(s) \leq 1$ and let $A \in K(X)$. If $h = S((\chi_A s)|Y)$ and $g = \chi_A s - h$, then $g = 0$ on Y and $\|g\|_p \leq 1$. Let $\mu \in \mathbb{K}, 0 < |\mu| < \epsilon/m_p(X)$. The set $V = \{x \in X : p(g(x)) > |\mu|\}$ is clopen and does not meet Y. Thus

$$\left|\int_V gdm\right| \leq m_p(V) \leq \epsilon, \quad \left|\int_{X\setminus V} gdm\right| \leq |\mu| m_p(X) \leq \epsilon.$$

Therefore $|m(A)s - \bar{m}(A)s| = |\int gdm| \leq \epsilon$. It follows that $\|m - \bar{m}\| \leq \epsilon|\lambda|$, which completes the proof.

3 The Completion of $(C_b(X, E), \beta_o)$

Let $C_{b,k}(X, E)$ be the space of all bounded E-valued functions on X whose restriction to every compact subset of X is continuous. For $p \in cs(E)$, let $\bar{\beta}_{o,p}$ be the locally convex topology on $C_{b,k}(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, h \in B_o(X)$. We define $\bar{\beta}_o$ to be the projective limit of the topologies $\bar{\beta}_{o,p}, p \in cs(E)$. For a sequence (K_n) of compact subsets of X and a sequence (d_n) of positive numbers, with $d_n \to \infty$, we denote by $W_{k,p}(K_n, d_n)$ the set $\bigcap_{n=1}^{\infty} \{f \in C_{b,k}(X, E) : \|f\|_{K_n,p} \leq d_n\}$. As in the case of β_o (see [7], p. 193), it can be shown that each $W_{k,p}(K_n, d_n)$ is a $\bar{\beta}_{o,p}$-neighborhood of zero. We also have the following Theorem whose proof is analogous to the proof of Proposition 2.6 in [7].

Theorem 3.1 *The sets of the form $W_{k,p}(K_n, |\lambda_n|)$, where (K_n) is an increasing sequence of compact subsets of X and (λ_n) a sequence in $\mathbb{K}$ with $0 < |\lambda_n| < |\lambda_{n+1}| \to \infty$, form a base at zero for $\bar{\beta}_{o,p}$.*

Theorem 3.2 *Let $p \in cs(E)$ and let W be an absolutely convex subset of $C_{b,k}(X, E)$. Then*

(1). If W is a $\bar{\beta}_{o,p}$-neighborhood of zero, then for every $r > 0$ there exist a compact subset Y of X and $\epsilon > 0$ such that

$$\{f \in C_{b,k}(X, E) : \|f\|_p \leq r, \|f\|_{Y,p} \leq \epsilon\} \subset W.$$

(2). If E is complete and polar and p a polar seminorm, then the converse holds in (1).

Proof: (1). It follows from the preceding Theorem.

(2). Assume that E is complete and polar, p is a polar seminorm and the condition holds in (1). Then, given $|\lambda| > 1$, there exist an increasing sequence (K_n) of compact subsets of X and a decreasing sequence (ϵ_n) of positive numbers such that $V_n \cap \lambda^n V \subset W$, where

$$V_n = \{f \in C_{b,k}(X, E) : \|f\|_{K_n,p} \leq \epsilon_n\}, V = \{f \in C_{b,k}(X, E) : \|f\|_p \leq 1\}.$$

Set $W_1 = V_1 \cap [\cap_{n=1}^{\infty}(V_{n+1} + \lambda^n V)]$. As in the proof of Theorem 2.8 in [7], we have that $W_1 \subset W$. Let now $\lambda_1 \in \mathbb{K}, 0 < |\lambda_1| < \min\{1, \epsilon_1\}$ and let $\lambda_n = \lambda^{n-1}$ for $n > 1$. We will finish the proof by showing that $W_2 = W_{k,p}(K_n, |\lambda_n|) \subset W_1$. So let $f \in W_2$. Then $f \in V_1$. Let m be a positive integer. There exists a linear map $T : C(K_{m+1}, E) \to C_{rc}(X, E)$ such that, for every $g \in C(K_{m+1}, E), Tg$ is an extension of g and $\|Tg\|_q = \|g\|_q$ for every polar $q \in cs(E)$. Let $g = T(f|K_{m+1}), h = f - g$. Then $h = 0$ on K_{m+1} and so $h \in V_{m+1}$. Also $\|g\|_p = \|f|K_{m+1}\|_p \le |\lambda|^m$ and so $f \in V_{m+1} + \lambda^m V$, which proves that $f \in W_1$. This clearly completes the proof.

In the following Theorem, for each $p \in cs(E)$, we will denote also by p the unique continuous extension of p to all of $\hat{E}$.

Theorem 3.3 *If E is polar, then $(C_{b,k}(X, \hat{E}), \bar{\beta}_o)$ coincides with the completion of $(C_b(X, E), \beta_o)$.*

Proof: Claim I: $C_b(X, E)$ is β_o-dense in $C_b(X, \hat{E})$. Indeed, let W be a convex β_o-neighborhood of zero in $C_b(X, \hat{E})$. Since β_o is coarser than τ_u, there exists $p \in cs(E)$ such that $W_1 = \{f \in C_b(X, \hat{E}) : \|f\|_p \le 1\} \subset W$. Let $A \in K(X)$ and $s \in \hat{E}$. Choose $w \in E$ with $p(s - w) < 1$. Then $\chi_A s - \chi_A w \in W_1$, which proves that $\chi_A s$ belongs to the closure of $C_b(X, E)$ in $C_b(X, \hat{E})$. Since the space spanned by the functions $\chi_A s, A \in K(X), s \in \hat{E}$, is β_o-dense in $C_b(X, \hat{E})$, our claim follows.
Let now W be a convex $\bar{\beta}_o$-neighborhood of zero in $C_{b,k}(X, \hat{E})$ and let $f \in C_{b,k}(X, \hat{E})$. There exists a polar continuous seminorm p on E such that W is a $\bar{\beta}_{o,p}$-neighborhood. In view of the preceding Theorem, there exist a compact subset Y of X and $\epsilon > 0$ such that

$$\{g \in C_{b,k}(X, \hat{E}) : \|g\|_p \le \|f\|_p, \|g\|_{Y,p} \le \epsilon\} \subset W.$$

Let $h \in C_b(X, \hat{E})$ be an extension of $f|Y$ such that $\|h\|_p = \|f\|_{Y,p}$. Now $\|f - h\|_p \le \|f\|_p$ and $f = h$ on Y, which implies that $f - h$ is in W. Thus $C_b(X, \hat{E})$ is $\bar{\beta}_o$-dense in $C_{b,k}(X, \hat{E})$, which, combined with Claim I, implies that $C_b(X, E)$ is $\bar{\beta}_o$-dense in $C_{b,k}(X, \hat{E})$.
Claim II: $(C_{b,k}(X, \hat{E}), \bar{\beta}_o)$ is complete. In fact, let (f_δ) be a $\bar{\beta}_o$-Cauchy net. For each $x \in X, (f_\delta(x))$ is a Cauchy net in $\hat{E}$. Thus we get a function $f : X \to \hat{E}, f(x) = \lim f_\delta(x)$. Since $f_\delta \to f$ uniformly on compact subsets of X, it follows that $f|Y$ is continuous for every compact set Y. Also, f is bounded. Indeed, suppose that there exist $p \in cs(E)$ and a sequence (x_n) of elements of X such that $p(f(x_n)) < p(f(x_{n+1})) \to \infty$. The set $W = \{g \in C_{b,k}(X, \hat{E}) : p(g(x_n)) \le p(f(x_n))/2\}$ is a $\bar{\beta}_{o,p}$-neighborhood of zero. Thus, there exists δ_o such that $f_\delta - f_{\delta_o} \in W$ for $\delta \ge \delta_o$. It follows from this that $p(f(x_n) - f_{\delta_o}(x_n)) \le p(f(x_n))/2$. Thus $p(f_{\delta_o}(x_n)) = p(f(x_n)) \to \infty$, a contradiction. By the above $f \in C_{b,k}(X, \hat{E})$. Moreover $f_\delta \to f$ in $C_{b,k}(X, \hat{E})$, which completes the proof.

Corollary 3.4 *If E is polar, then $(C_b(X, E), \beta_o)$ is complete iff E is complete and every bounded E-valued f on X such that $f|Y$ is continuous, for every compact subset Y of X, is continuous on X.*

Theorem 3.5 *If E is polar and complete, then $(C_b(X,E),\beta_o)$ is complete iff it is quasicomplete.*

Proof: Assume that $(C_b(X,E),\beta_o)$ is quasicomplete and let $f \in C_{b,k}(X,E)$. For each compact subset K of X there exists f_K in $C_b(X,E)$ such that $f_K = f$ on K and $\|f\|_{K,p} = \|f_K\|_p$ for each continuous polar seminorm p on E. The set $\{f_K : K \subset X, K\text{compact}\}$ is contained in the uniformly bounded subset D of $C_b(X,E)$ consisting of all g with $\|g\|_p \leq \|f\|_p$ for all $p \in cs(E)$, p polar. On D, β_o coincides with the topology τ_k of compact convergence. Ordering the family $\mathcal{K}$ of all compact subsets of X by set inclusion, we get a net $(f_K)_{K\in\mathcal{K}}$ in $C_b(X,E)$ which is τ_k-Cauchy and hence β_o-Cauchy. Since D is β_o-bounded, there exists $g \in C_b(X,E)$ such that the net (f_K) is β_o-cnvergent to g. But then $g(x) = \lim f_K(x) = f(x)$ for all x and so $f = g \in C_b(X,E)$. Now the result follows from the preceding Corollary.

Recall that a topological space Y is called a P-space if every zero set is open. In case Y is zero-dimensional, Y is a P-space iff every $\mathbb{K}$-zero set is open, equivalently iff every countable intersection of clopen sets is clopen.

Theorem 3.6 *If X is a P-space, then $(C_b(X,E),\beta_o)$ is sequentially-complete iff E is sequentially-complete.*

Proof: Assume that $(C_b(X,E),\beta_o)$ is sequentially-complete and let (s_n) be a Cauchy sequence in E. The sequence $(g_n), g_n(x) = s_n$ for all $x \in X$, is β_o-Cauchy. If (g_n) is β_o-cnvergent to g, then $g(x) = \lim s_n$ and so E is sequentially-complete. Conversely, let E be sequentially-complete and let (f_n) be a β_o-Cauchy sequence in $C_b(X,E)$. Since β_o is finer than the topology of simple convergence, the limit $f(x) = \lim f_n(x)$ exists in E for each $x \in X$. Then f is bounded. Indeed, assume that there exists a $p \in cs(E)$ such that $\|f\|_p = \infty$. Choose a sequence (a_n) of elements of X such that $p(f(a_n)) > n$ for all n. The set

$$W = \{g \in C_b(X,E) : p(g(a_n)) \leq n, n \in \mathbb{N}\}$$

is a β_o-neighborhood of zero. Let n_o be such that $f_n - f_{n_o} \in W$ for $n \geq n_o$. For $n \geq n_o$ we have that $p(f_n(a_k) - f_{n_o}(a_k)) \leq k$ and so $p(f(a_k) - f_{n_o}(a_k)) \leq k$, which implies that $p(f_{n_o}(a_k)) = p(f(a_k)) > k$, for all k, a contradiction since f_{n_o} is bounded. Also f is continuous. In fact, let $x \in X$ and let D be a clopen neighborhood of $f(x)$ in E. Each $f_n^{-1}(D)$ is a clopen neighborhood of x and so $V = \cap V_n$ is a neighborhood of x since X is a P-space. Moreover, for $y \in V$, $f(y) \in \bar{D} = D$, which proves the continuity of f at x. Moreover, since β_o has a base at zero consisting of sets which are closed with respect to the topology of simple convergence, it follows that (f_n) is β_o-convegent to f, and this completes the proof.

4 Product Measures

Let $B_{ou}(X)$ be the family of all $\phi \in B_o(X)$ for which $|\phi|$ is upper semicontinuous. As it is shown in [12], if $|\lambda| > 1$, then for every $\phi \in B_o(X)$ there exists $\psi \in B_{ou}(X)$ such that $|\psi| \leq |\phi| \leq \lambda\psi|$. Thus β_o is defined by the seminorms $f \mapsto \|\phi f\|, \phi \in B_{ou}(X), p \in cs(E)$. If Y is another Hausdorff zero-dimensional topological space, then for each $\phi_1 \in B_{ou}(X)$ and each $\phi_2 \in B_{ou}(Y)$, the function $\phi_1 \times \phi_2$, which is defined on $X \times Y$ by $\phi_1 \times \phi_2(x,y) = \phi_1(x)\phi_2(y)$, is in $B_{ou}(X \times Y)$. Also, given $\phi \in B_{ou}(X \times Y)$, there exist $\phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y)$ such that $|\phi_1 \times \phi_2| \geq |\phi|$. Thus the topology β_o on $C_b(X,E)$ is defined by the seminorms $f \mapsto \sup_{x\in X, y\in Y} p(\phi_1(x)\phi_2(y)f(x,y))$, where $\phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y), p \in cs(E)$.

Theorem 4.1 *Let X, Y be zero-dimensional Hausdorff topological spaces. If G is the subspace of $C_b(X \times Y, E)$ spanned by the functions $\chi_{A\times B}s, A \in K(X), B \in K(Y), s \in E$, then G is β_o-dense in $C_b(X \times Y, E)$.*

Proof: Let $p \in cs(E), \phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y), W = \{f \in C_b(X \times Y, E) : p(\phi_1(x)\phi_2(y)f(x,y)) \leq 1\}$. Let $f \in C_b(X \times Y, E)$. The set

$$D = \{(x,y) : p(\phi_1(x)\phi_2(y)f(x,y)) \geq 1/2\}$$

is compact. If D_1, D_2 are the projections of D on X, Y, respectively, then $D \subset D_1 \times D_2$. Choose $d > \|\phi_1\|, \|\phi_2\|$ and let $x \in D_1$. There exists $y \in Y$ such that $(x,y) \in D$ and hence $\phi_1(x) \neq 0$. The set $Z_x = \{z \in X : |\phi_1(z)| < 2|\phi_1(x)|\}$ is open and contains x. Using the compactness of D_2, we find a clopen neighborhood A_x of x contained in Z_x such that $p(f(z,y) - f(x,y)) < 1/d^2$ for all $z \in A_x, y \in D_2$. Because of the compactness of D_1, there are $x_1, \ldots, x_m$ in D_1 such that $D_1 \subset \bigcup_{k=1}^m A_{x_k}$. Let $A_1 = A_{x_1}, A_{k+1} = A_{x_{k+1}} \setminus \bigcup_{i=1}^k A_{x_i}, k = 1, \ldots, m-1$. Keeping those of the A_k which are not empty, we may assume that each $A_k \neq \emptyset$. For each $1 \leq k \leq m$, there are pairwise disjoint clopen sets $B_{k1}, \ldots, B_{kn_k}$ of Y, covering D_2, and $y_{kj} \in B_{kj}$ such that $p(f(x_k, y) - f(x_k, y_{kj})) < 1/(2d^2)$ if $y \in B_{kj}$. Let now $h = \sum_{k=1}^m \sum_{j=1}^{n_k} \chi_{A_k \times B_{kj}} f(x_k, y_{kj})$. Then $h \in G$. Moreover, $p(\phi_1(x)\phi_2(y)(f(x,y) - h(x,y)) \leq 1$ for all (x,y). To prove this, we consider the three possible cases:

Case I. $x \notin \bigcup_{k=1}^m A_k$. Then $h(x,y) = 0$. Also $(x,y) \notin D$ and so $p(\phi_1(x)\phi_2(y)f(x,y)) \leq 1/2$.

Case II. $x \in A_k, y \in D_2$. There exists j with $y \in B_{kj}$. Now $p(f(x,y) - f(x_k,y)) < 1/d^2$ and $p(f(x_k,y) - f(x_k,y_{kj})) < \frac{1}{2d^2}$, which implies that $p(\phi_1(x)\phi_2(y)(f(x,y) - h(x,y)) \leq 1$.

Case III. $x \in A_k, y \notin D_2$. Then $(x,y) \notin D$ and so $p(\phi_1(x)\phi_2(y)f(x,y)) \leq 1/2$. If $h(x,y) \neq 0$, then $y \in B_{kj}$ for some j, and so $h(x,y) = f(x_k, y_{kj}), p(f(x_k,y) - f(x_k,y_{kj})) \leq \frac{1}{2d^2}$. Since $x \in A_{x_k}$, we have that $|\phi_1(x)| < 2|\phi_1(x_k)|$ and thus $p(\phi_1(x)\phi_2(y)f(x_k,y)) \leq 2p(\phi_1(x_k)\phi_2(y)f(x_k,y)) \leq 1$ since $(x_k, y) \notin D$. It follows that $p(\phi_1(x)\phi_2(y)h(x,y)) \leq 1$ and our claim follows. This clearly completes the proof.

Theorem 4.2 *If $\mu \in M_\tau(X)$ and $m \in M_{t,p}(Y, E')$, then there exists a unique $\bar{m} \in M_t(X \times Y, E')$ such that $\bar{m}(A \times B) = \mu(A)m(B)$ for each $A \in K(X)$ and each $B \in K(Y)$. Moreover, $\bar{m} \in M_{t,p}(X \times Y, E')$.*

Proof: By [12], Theorem 4.6, there exists a linear map

$$\omega : M = (C_b(X), \beta_o) \otimes (C_b(Y, E), \beta_o) \to (C_b(X \times Y, E), \beta_o)$$

such that $\omega(g \otimes f) = g \times f$, for all $g \in C_b(X), f \in C_b(Y, E)$, where $(g \times f)(x, y) = g(x)f(y)$, and $\omega : M \to \omega(M)$ is a topological isomorphism. In view of the preceding Theorem, $\omega(M)$ is β_o-dernse in $C_b(X \times Y, E)$. The bilinear map

$$T : (C_b(X), \beta_o) \times (C_b(Y, E), \beta_o) \to \mathbb{K}, \quad T(g, f) = \left(\int g d\mu\right)\left(\int f dm\right)$$

is continuous. Hence we have a continuous linear map $\phi : M \to \mathbb{K}, \phi(g \otimes f) = T(g, f)$. Since $\omega : M \to \omega(M)$ is a topological isomorphism, it follows that the linear map $\psi : \omega(M) \to \mathbb{K}, \psi = \phi \circ \omega^{-1}$, is β_o-continuous on $\omega(M)$. As $\omega(M)$ is β_o-dense in $C_b(X \times Y, E)$, there is a continuous extension $\tilde{\psi}$ of ψ to all of $C_b(X \times Y, E)$. Thus, there exists $\bar{m} \in M_t(X \times Y, E)$ such that $\tilde{\psi}(h) = \int h d\bar{m}$ for all $h \in C_b(X \times Y, E)$. In particular, for $g \in C_b(X), f \in C_b(Y, E)$, we have $\psi(g \times f) = \int (g \times f) d\bar{m} = (\int g d\mu)(\int f dm)$. If $A \in K(X), B \in K(Y), s \in E$ and $h = \chi_{A \times B} s = \chi_A \times (\chi_B s)$, then

$$\bar{m}(A \times B)s = \tilde{\psi}(h) = \mu(A)m(B)s$$

and so $\bar{m}(A \times B)) = \mu(A)m(B)$.

Let now $m_1 \in M_t(X \times Y, E')$ be such that $m_1(A \times B)) = \mu(A)m(B)$ for all $A \in K(X), B \in K(Y)$. Consider the β_o-continuous linear forms $\phi_1(h) = \int h d\bar{m}, \phi_2(h) = \int h dm_1$. If G is as in the proof of the preceding Theorem, then $\phi_1 = \phi_2$ on G and hence $\phi_1 = \phi_2$ since G is β_o-dense in $C_b(X \times Y, E)$. Thus $\bar{m} = m_1$. Finally, assume that $m \in M_{t,p}(X, E')$. There are $\phi_1 \in B_{ou}(X)$ and $\phi_2 \in B_{ou}(Y)$ such that $|\int g dm| \leq \|\phi_1 g\|$ and $|\int f dm| \leq \|\phi_2 f\|_p$ for all $g \in C_b(X), f \in C_b(Y, E)$. Thus, for $h = g \times f$, we have that $|\int h d\bar{m}| \leq \|(\phi_1 \times \phi_2)h\|_p$. Since the map $h \mapsto \|(\phi_1 \times \phi_2)h\|_p$ is a β_o-continuous seminorm on $C_b(X \times Y, E)$, it follows that $|\int h d\bar{m}| \leq \|(\phi_1 \times \phi_2)h\|_p$ for all $h \in C_b(X \times Y, E)$. In particular, for $D \in K(X \times Y)$ and $s \in E$, we have

$$|\bar{m}(D)s| \leq p(s) \sup_{(x,y) \in X \times Y} |\phi_1(x)\phi_2(y)| \leq p(s)\|\phi_1 \times \phi_2\|.$$

Thus, $\bar{m}_p(X \times Y) \leq \|\phi_1 \times \phi_2\| = \|\phi_1\|\|\phi_2\|$. This completes the proof.

Definition 4.3 *For $\mu \in M_\tau(X)$ and $m \in M_t(Y, E')$, we define by $\mu \times m$ the unique element $\bar{m}$ of $M_t(X \times Y, E')$ for which $\bar{m}(A \times B) = \mu(A)m)B)$ for $A \in K(X), B \in K(Y)$. We call this $\bar{m}$ the product of μ and m.*

Theorem 4.4 *Let $h \in C_b(X \times Y, E)$ and $m \in M_{t,p}(Y, E')$. Then the function*

$$g : X \to \mathbb{K}, \quad g(x) = \int_Y f(x, y) dm(y)$$

is bounded and continuous.

Proof: Without loss of generality, we may assume that $\|m\|_p \leq 1$ and $\|f\|_p \leq 1$. Let $\epsilon > 0$ and let D be a compact subset of Y such that $m_p(A) < \epsilon$ if A is disjoint from D. Let $x_o \in X$. For each $y \in D$ there are clopen neighborhoods V_y and W_y of y and x_o, respectively, such that $p(f(x, z) - f(x_o, y)) < \epsilon$ if $x \in W_y, z \in V_y$. Let $y_1, \cdots, y_n$ in D be such that $D \subset V = \bigcup_{k=1}^n V_{y_k}$ and let $W = \bigcap_{k=1}^n W_{y_k}$. Then, for

$x \in W, y \in V$ we have that $p(f(x,y) - f(x_o,y)) \leq \epsilon$. It follows that , for $x \in W$, we have

$$|\int_V f(x,y)dm(y) - \int_V f(x_o,y)dm(y)| \leq \epsilon.$$

Also,

$$|\int_{Y\setminus V} f(x,y)dm(y)| \leq \|f\|_p m_p(Y \setminus V) \leq \epsilon \ \ and \ \ |\int_{Y\setminus V} f(x_o,y)dm(y)| \leq \epsilon.$$

Thus, for $x \in W$, we have $|g(x) - g(x_o)| \leq \epsilon$, which proves thast g is continuous. Moreover $\|g\| \leq 1$.

Theorem 4.5 *Let $\mu \in M_\tau(X), m \in M_{t,p}(Y,E'), \bar{m} = \mu \times m$. If $h \in C_b(X \times Y, E)$, then $\int h d\bar{m} = \int_X [\int_Y h(x,y)dm(y)]d\mu(x)$.*

Proof: Define

$$\psi : C_b(X \times Y, E) \to \mathbb{K}, \quad \psi(f) = \int_X \int_Y f(x,y)dm(y)d\mu(x).$$

There are $\phi_1 \in B_{ou}(X), \phi_2 \in B_{ou}(Y)$ such that for every $g \in C_b(X)$ and every $f \in C_b(Y,E)$, we have

$$|\int g d\mu| \leq \|\phi_1 g\| \quad and \quad |\int f dm| \leq \|\phi_2 f\|_p.$$

Now, for all $x \in X$, we have $|\int_Y h(x,y)dm(y)| \leq \sup_{y\in Y} |\phi_2(y)|p(h(x,y))$ and

$$|\int_X [\int_Y h(x,y)dm(y)]d\mu(x)| \leq \sup_{x\in X}[\sup_{y\in Y} |\phi_2(y)|p(h(x,y))]|\phi_1(x)| = \sup_{(x,y)\in X\times Y} |\phi_1 \times \phi_2(x,y)|p(h(x,y)).$$

Since $\phi_1 \times \phi_2 \in B_{ou}(X \times Y)$, it follows that ψ is β_o-continuous on $C_b(X \times Y, E)$. For $A \in K(X), B \in K(Y), f = \chi_{A\times B)}s = \chi_A \times (\chi_B s)$, we have

$$\psi(f) = \int_X [\int_Y \chi_A(x)\chi_B(y)dm(y)]d\mu(x) = \mu(A)m(B)s$$

and $\int f d\bar{m} = \mu(A)m(B)s$. Thus $\psi(f) = \int f d\bar{m}$ for $f \in G$, where G is as in Theorem 4.1. Since G is β_o-dense in $C_b(X \times Y, E)$, we have that $\psi(f) = \int f d\bar{m}$ for all $f \in C_b(X \times Y, E)$. This completes the proof.

5 (VR)-Integrals

Van Rooij defined in [16] integration of functions in $\mathbb{K}^X$ with respect to members μ of $M_\tau(X)$. His definition however cannot be applied for arbitrary μ in $M(X)$. Let $\mu \in M_\tau(X)$. He defined $N_\mu : X \to \mathbb{R}$ by $N_\mu(x) = \inf\{|\mu|(A) : x \in A \in K(X)\}$. Then N_μ is upper semicontinuous and, for every $\epsilon > 0$, the set $\{x \in X : N_\mu(x) \geq \epsilon\}$ is compact. For $A \in K(X)$ we have that $|\mu|(A) = \sup_{x\in A} N_\mu(x)$. For $f \in \mathbb{K}^X$, he defined $\|f\|_{N_\mu} = \sup_x |f(x)|N_\mu(x)$. If g is a $K(X)$-simple function, i.e. $g =$

$\sum_{k=1}^{n} \alpha_k \chi_{A_k}$, with $A_k \in K(X), \alpha_k \in \mathbb{K}$, he defined $\int g d\mu = \sum_{k=1}^{n} \alpha_k m(A_k)$. Van Rooij called an $f \in \mathbb{K}^X$ μ-integrable if there exists a sequence (g_n) of simple functions such that $\|f - g_n\|_{N_\mu} \to 0$. In this case, he called integral of f the $\lim \int g_n d\mu$. We will denote by $(VR) \int f d\mu$ the integral of f in his sense. It was proved in [10] that, for $\mu \in M_\tau(X)$, if f is μ-integrable in our sense, then f is also integrable in Van Rooij's sense and the two integrals coincide.

In this section we will assume that E is a normed space and we will define the integral $(VR) \int f dm$ of an f in E^X with respect to an $m \in M_t(X, E') = M_\tau(X, E')$. Most of the aguments we will use will be analogous to the ones used in [16] where scalar-valued measurers and functions in $\mathbb{K}^X$ are treated. Let $m \in M_t(X, E')$. As in [16], we define

$$N_m : X \to \mathbb{R}, N_m(x) = \inf\{|m|(A) : x \in A \in K(X)\},$$

where $|m| = m_{\|.\|}$. Then N_m is upper-semicontinuous and $|m|(A) = \sup_{x \in A} N_m(x)$ for each $A \in K(X)$.

Let $S(X, E)$ be the space of all E-valued $K(X)$-simple functions on X. For $h \in E^X$, we define $\|h\|_{N_m} = \sup_{x \in X} N_m(x) \|h(x)\|$.

Lemma 5.1 *If $m \in M_t(X, E')$ and $g = \sum_{k=1}^{n} \chi_{A_k} s_k \in S(X, E)$, then*

$$|\sum_{k=1}^{n} m(A_k) s_k| \leq \|g\|_{N_m} \leq \|g\| \|m\|.$$

Proof: Without loss of generality we may assume that the sets $A_1, \dots, A_n$ are pairwise disjoint. Since, for $A \in K(X)$ and $s \in E$, we have $|m(A)s| \leq \|s\| |m|(A) = \|s\| \sup_{x \in A} N_m(x)$, the Lemma follows.

We have the following easily established

Lemma 5.2 *Let $m \in M_t(X, E')$ and $f \in E^X$. Assume that there exists a sequence $(g_n) \subset S(X, E)$ such that $\|f - g_n\|_{N_m} \to 0$. Then: (1) The $\lim_{n \to \infty} \int g_n dm$ exists.*
(2) If (h_n) is another sequence in $S(X, E)$ such that $\|f - h_n\|_{N_m} \to 0$, then $\lim_{n \to \infty} \int g_n dm = \lim_{n \to \infty} \int h_n dm$.
(3) $|\lim_{n \to \infty} \int g_n dm| \leq \|f\|_{N_m} < \infty$.

Definition 5.3 *Let $m \in M_t(X, E')$. A function $f \in E^X$ is called (VR)-integrable with respect to m if there exists a sequence $(g_n) \subset S(X, E)$ such that $\|f - g_n\|_{N_m} \to 0$. In this case we define*

$$(VR) \int f dm = \lim_{n \to \infty} \int g_n dm.$$

Let now $m \in M_t(X, E')$ and let

$$\mathcal{S}_m = \{A \subset X : \chi_A s \text{ is (VR)-integrable for all } s \in E\}.$$

As in [16], Lemma 7.3, we have the following

Lemma 5.4 *Let $m \in M_t(X, E')$ and $A \subset X$. Then $A \in \mathcal{S}_m$ iff, for every $\epsilon > 0$, there exists $B \in K(X)$ such that $N_m < \epsilon$ on $A \Delta B = (A \setminus B) \cup (B \setminus A)$.*

Proof: Assume that $A \in \mathcal{S}_m$ and let s be a non-zero element of E. Let $g \in S(X, E)$ be such that $\|\chi_A s - g\|_{N_m} < \epsilon\|s\|$. If $B = \{x : \|g(x) - s\| < \|s\|\}$, then $B \in K(X)$ and $\|g(x) - \chi_B(x)s\| \leq \min\{\|g(x)\|, \|g(x) - s\|\} \leq \|g(x) - \chi_A(x)s\|$ and so

$$\|\chi_A s - \chi_B s\|_{N_m} \leq \max\{\|\chi_A s - g\|_{N_m}, \|\chi_B s - g\|_{N_m}\} = \|\chi_A s - g\|_{N_m} < \epsilon\|s|,$$

which implies that $N_m < \epsilon$ on $A\Delta B$.
Conversely, suppose that the condition is satisfied and let s be a non-zero element of E. Choose $B \in K(X)$ such that $N_m < \epsilon/\|s\|$ on $A\Delta B$. Then $\|\chi_A s - \chi_B s\|_{N_m} \leq \epsilon$ which completes the proof.

We can easily prove the following

Lemma 5.5 *Let $m \in M_t(X, E')$. Then: (1) For each $A \in \mathcal{S}_m$, the complement A^c is also in $\mathcal{S}_m$.*
(2) If $A_1, A_2 \in \mathcal{S}_m$, then $A_1 \cup A_2$ and $A_1 \cap A_2$ are in $\mathcal{S}_m$.
(3) $K(X) \subset \mathcal{S}_m$.
(4) $A \in \mathcal{S}_m$ iff, for each $\epsilon > 0$, there exists $B \in K(X)$ such that $A \cap X_{m,\epsilon} = B \cap X_{m,\epsilon}$, where $X_{m,\epsilon} = \{x : N_m(x) \geq \epsilon\}$.

For $m \in M_t(X, E')$, we denote by τ_m the zero-dimensional topology on X having $\mathcal{S}_m$ as a base. Clearly τ_m is finer than the topology τ of X. We denote by X_m the set X equipped with the topology τ_m.

Theorem 5.6 *Let $m \in M_t(X, E')$. Then $X_{m,\epsilon}$ is τ_m-compact for each $\epsilon > 0$.*

Proof: It suffices to show that every cover $\mathcal{U}$ of $X_{m,\epsilon}$ by sets in $\mathcal{S}_m$ has a finite subcover. Without loss of generality, we may assume that $A_1 \cup A_2$ is in $\mathcal{U}$ if $A_1, A_2 \in \mathcal{U}$. Since N_m is τ_m-upper semicontinuous, $X_{m,\epsilon}$ is τ_m-closed. Hence the family

$$\mathcal{V} = \{(V \cup Z)^c : V \in \mathcal{U}, Z \subset X^c_{m,\epsilon}, Z \in \mathcal{S}_m\}$$

is downwards directed to the empty set. Since $|m|$ is τ-additive, there exist $V \in \mathcal{U}, Z \subset X^c_{m,\epsilon}$ such that $|m|((V \cup Z)^c) < \epsilon$ and so $X_{m,\epsilon} \subset V \cup Z$, which implies that $X_{m,\epsilon} \subset V$, and we are done.

Since $X_{m,\epsilon}$ is τ_m-compact and τ is Hausdorff, it follows that $\tau = \tau_m$ on $X_{m,\epsilon}$.

Lemma 5.7 *For $m \in M_t(X, E')$, an $A \subset X$ is τ_m-clopen iff it is in $\mathcal{S}_m$.*

Proof: Asuume that A is τ_m-clopen. Then, for $\epsilon > 0$, the set $A \cap X_{m,\epsilon}$ is clopen in in $X_{m,\epsilon}$ for the topology induced by τ_m and hence for the topology induced by τ. Since $X_{m,\epsilon}$ is τ-compact, there exists $B \in K(X)$ such that $A \cap X_{m,\epsilon} = B \cap X_{m,\epsilon}$. The result now follows from Lemma 5.5.

Proposition 5.8 *If $m \in M_t(X, E')$ and $f \in E^X$, then f is τ_m-continuous iff $f|X_{m,\epsilon}$ is τ-continuous for each $\epsilon > 0$.*

Proof: Since $\tau = \tau_m$ on $X_{m,\epsilon}$, the necessity is clear. Conversely, assume that the condition is satisfied. If D is a clopen subset of E, then $f^{-1}(D) \cap X_{m,\epsilon}$ is clopen in $X_{m,\epsilon}$ for the topology induced on $X_{m,\epsilon}$ by τ. Since $X_{m,\epsilon}$ is τ-compact, there exists $A \in K(X)$ such that $A \cap X_{m,\epsilon} = f^{-1}(D) \cap X_{m,\epsilon}$. Thus $f^{-1}(D$ is τ_m-clopen by Lemma 5.5 and the result follows.

Theorem 5.9 *Let $m \in M_t(X, E')$. For a τ_m-clopen subset A of X, we define $\bar{m}(A)$ on E by $\bar{m}(A)s = (VR)\int \chi_A s dm$. Then : 1) $\bar{m}(A) \in E'$.*
2) $\bar{m}(A) \in M_t(X_m, E')$, $\|m| = \|\bar{m}\|$ and $|\bar{m}|(A) = |m|(A)$ for $A \in K(X)$.

Proof: 1) It follows from the inequality

$$|(VR)\int \chi_A s dm| \leq \sup_{x\in A} \|s\| N_m(x) \leq \|m\|\|s\|.$$

2) Clearly $\bar{m}$ is finitely additive. Let $\mathcal{A}$ be a family of τ_m-clopen sets which is downwards directed to the empty set and let $Y = X_{m,\epsilon}$. For each $A \in \mathcal{A}$, there exists $B \in K(X)$ such that $A \cap Y = B \cap Y$. Let

$$\mathcal{B} = \{B \in K(X) : \exists A \in \mathcal{A}, A \cap Y = B \cap Y\}.$$

Let $B_1, B_2 \in \mathcal{B}$ and let $A_1, A_2 \in \mathcal{A}$ such that $A_i \cap Y = B_i \cap Y$, for $i = 1, 2$. Let $A \in \mathcal{A}, A \subset A_1 \cap A_2$ and choose $B \in K(X)$ with $A \cap Y = B \cap Y$. If $D = A \cap B_1 \cap B_2$, then $A \cap Y = D \cap Y$ and so $D \in \mathcal{B}$, which proves that $\mathcal{B}$ is downwards directed. Moreover $\bigcap \mathcal{B} = \emptyset$. Indeed assume that $x \in \bigcap \mathcal{B}$. If $x \notin Y$, then there exists $Z \in K(X)$ containing x with $|m|(Z) < \epsilon$ and so Z is disjoint from Y. If $B \in \mathcal{B}$, then there exists $A \in \mathcal{A}$ with $A \cap Y = B \cap Y = (B \setminus Z) \cap Y$ and so $B \setminus Z \in \mathcal{B}$, a contradiction since $x \notin B \setminus Z$. Thus x must be in Y and so $x \in \cap \mathcal{B} = \bigcap_{B\in\mathcal{B}} B \cap Y$. Given $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ with $A \cap Y = B \cap Y$ and so $x \in A$, i.e. $x \in \bigcap \mathcal{A}$, a contradiction. Thus $\mathcal{B}$ is downwards directed to the empty set. Since $m \in M_t(X, E')$, there exists $B \in \mathcal{B}$ with $|m|(B) < \epsilon$. Let $A \in \mathcal{A}$ with $A \cap Y = B \cap Y = \emptyset$. If $x \in A$, then $x \notin Y$ and so $N_m(X) < \epsilon$. If G is a τ_m-clopen set contained in A, then for each $s \in E$ we have

$$|\bar{m}|(G)s| \leq \sup_{x\in G} \|s\| N_m(x) \leq \epsilon \|s|$$

and so $|\bar{m}|(A) \leq \epsilon$. This proves that $\bar{m} \in M_\tau(X_m, E') = M_t(X_m, E')$. Finally, let $A \in K(X)$. Clearly $|m|(A) \leq |\bar{m}|(A)$. On the other hand, let D be a τ_m-clopen subset of A. For each $s \in E$, we have

$$|\bar{m}(D)s| = |(VR)\int \chi_D s dm| \leq \sup_{x\in D} \|s\| N_m(x) \leq \|s\| |m|(A),$$

which proves that $|m|(A) \geq |\bar{m}|(A)$, and the result follows.

Proposition 5.10 *If $m \in M_t(X, E')$, then $N_{\bar{m}} = N_m$.*

Proof: Since $|m|(A) = |\bar{m}|(A)$ for $A \in K(X)$, it follows that $N_{\bar{m}} \leq N_m$. Assume that, for some $x \in X$, we have $N_{\bar{m}}(x) < \epsilon < N_m(x)$. There exists a τ_m-clopen set A containing x with $|\bar{m}|(A) < \epsilon$. Let $B \in K(X)$ such that $A \cap Y = B \cap Y, Y = \{y :$

$N_m(y) \geq \epsilon\}$. Then $x \in B$ and so $|m|(B) \geq N_m(x) > \epsilon$. Let $D \in K(X)$ contained in B and $s \in E$ be such that $|m(D)s|/\|s\| > \epsilon$. Then $|\bar{m}(D \cap A)s|/\|s\| \leq |\bar{m}|(D \cap A) < \epsilon$. Since $m(D) = \bar{m}(D)$, we have that $|m(D)s| = |m(D)s - \bar{m}(D \cap A)s| = |\bar{m}(D \setminus A)s| \leq \|s\| \sup_{y \in D \setminus A} N_m(y)$. But, if $y \in D \setminus A$, then $N_m(y) < \epsilon$, since $D \subset B$ and $A \cap Y = B \cap Y$, and so $|m(D)s| \leq \epsilon \|s\|$, a contradiction. This completes the proof.

Lemma 5.11 *Let $m \in M_t(X, E')$ and $g \in S(X_m, E)$. Then, for each $\epsilon > 0$, there exists $h \in S(X, E)$ such that $\|g - h\|_{N_m} \leq \epsilon$.*

Proof: If $g \neq 0$, there are paiwise disjoint τ_m-clopen sets $A_1, \ldots, A_n$ and non-zero elements $s_1, \ldots, s_n$ in E such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Let $\alpha = \min\{\|s_i\| : i = 1, \ldots, n\}$. For each i, choose $B_i \in K(X)$ with $N_m < \epsilon/\alpha$ on $A_i \Delta B_i$. Let $Z_1 = B_1, Z_{k+1} = B_{k+1} \setminus \bigcup_{i=1}^k B_i$, for $k = 1, \ldots, n-1$. Then $N_m < \epsilon/\alpha$ on $A_i \Delta Z_i$. Let $h = \sum_{k=1}^n \chi_{Z_k} s_k$. Since $x \in \bigcup_{k=1}^n A_k \Delta Z_k$ when $g(x) \neq h(x)$, we have that $\|g - h\|_{N_m} \leq \epsilon$ and the result follows.

Corollary 5.12 *If $m \in M_t(X, E')$ and $f \in E^X$, then f is (VR)-integrable with respect to m iff it is (VR)-integrable with respect to $\bar{m}$.In this case we have $(VR) \int f dm = (VR) \int f d\bar{m}$.*

Theorem 5.13 *For $m \in M_t(X, E')$ and $f \in E^X$ the following are equivalent:*
(1) f is (VR)-integrable with respect to m.
(2) For each $\epsilon > 0$, $f|X_{m,\epsilon}$ is continuous and the set $D = \{x : \|f(x)\| N_m(x) \geq \epsilon\}$ is τ_m-compact.

Proof: (1) $\Rightarrow$ (2). Choose $g \in S(X, E)$ such that $\|f - g\|_{N_m} < \epsilon^2$. Let $x_o \in X_{m,\epsilon}$ and $V = \{x : \|g(x) - g(x_o)\| < \epsilon\}$. If $x \in V \cap X_{m,\epsilon}$, then $\|f(x) - g(x)\| \leq \epsilon$ and so $\|f(x) - f(x_o)\| \leq \epsilon$, which proves that $f|X_{m,\epsilon}$ is continuous. To prove that D is τ_m-compact, choose $g \in S(X, E)$ with $\|g - f\|_{N_m} < \epsilon$. Then

$$\{x : \|f(x)\| N_m(x) \geq \epsilon\} = \{x : \|g(x)\| N_m(x) \geq \epsilon\}.$$

Let $A_1, \ldots, A_n \in K(X)$ be disjoint and s_i non-zero elements of E such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Then

$$A_k \bigcap \{x : \|g(x)\| N_m(x) \geq \epsilon\} = \{x : \|s_k\| N_m(x) \geq \epsilon\} = A_k \bigcap \{x : N_m(x) \geq \epsilon/\|s_k\|\} = D_k.$$

Thus $D = \bigcup D_k$ is τ_m-compact.
(2) $\Rightarrow$ (1). Our hypothesis implies (in view of Proposition 5.8) that f is τ_m-continuous. Since D is τ_m-compact and N_m is τ_m-upper semicontinuous, there exists a positive number α such that $N_m(x) < \alpha$ for each $x \in D$. For each $x \in D$, the set $M_x = \{y : \|f(y) - f(x)\| < \epsilon/\alpha\}$ is a τ_m-clopen neighborhood of x. If $M_x \cap M_y \neq \emptyset$, then $M_x = M_y$. Hence there are $a_1, \ldots, a_n \in D$ such that the sets M_{a_k} are disjoint and cover D. Let $0 < \epsilon_1 < \alpha$ be such that $\|f(a_k\|\epsilon_1 < \epsilon$, for $k = 1, \ldots, n$. There are $A_k \in K(X)$ such that $M_{a_k} \cap Y = A_k \cap Y$, where $Y = \{x : N_m(x) \geq \epsilon_1\}$. Take $Z_1 = A_1, Z_{k+1} = B_{k+1} \setminus \bigcup_{i=1}^k A_i$, for $k = 1, \ldots, n-1$. Then $Z_k \cap Y = A_k \cap Y$. Let $g = \sum_{k=1}^n \chi_{A_k} f(a_k)$. Then $\|f(x) - g(x)\| N_m(x) \leq \epsilon$ for all x. To show this, we consider the two possible cases. Case I: $x \in D$. Then $x \in M_{a_k}$, for some k,

and so $\|f(x) - f(a_k)\|N_m(x) \le \alpha\|f(x) - f(a_k)\| < \epsilon$. Since $\|f(x)\|N_m(x) \ge \epsilon$, we have $\|f(x)\| = \|f(a_k)\|$. If now $x \in Y$, then $x \in Z_k$ and so $g(x) = f(a_k)$, which implies that $\|f(x) - g(x)\|N_m(x) = \|f(x) - f(a_k)\|N_m(x) < \epsilon$. If $x \notin Y$, then $\|f(x)\|N_m(x) = \|f(a_k)\|N_m(x) \le \epsilon_1\|f(a_k)\| < \epsilon$, a contradiction.
Case II: $x \notin D$. Then $\|f(x)\|N_m(x) < \epsilon$. If $\|f(x) - g(x)\|N_m(x) > \epsilon$, then $\|g(x)\|N_m(x) > \epsilon$ and so $x \in Z_k$, for some k, which implies that $g(x) = f(a_k)$ and so $\|f(a_k)\|N_m(x) > \epsilon$. Consequently, $N_m(x) > \epsilon_1$ and thus $x \in Z_k \cap Y = M_{a_k} \cap Y$. But then

$$\|f(x) - g(x)\|N_m(x) = \|f(x) - f(a_k)\|N_m(x) < \epsilon_1\epsilon/\alpha < \epsilon,$$

a contradiction. Thus $\|f - g\|_{N_m} \le \epsilon$ which proves that f is (VR)-integrable with respect to m and we are done.

Lemma 5.14 *If $\phi \in E'$ and Y a compact subset of X, then there exists an $m \in M_t(X, E')$ such that $N_m(x) = \|\phi\|$ for $x \in Y$ and $N_m(x) = 0$ for $x \notin Y$.*

Proof: By [16], p. 273, thete exists a $\mu \in M_\tau(X)$ such that $N_\mu(x) = 1$ for $x \in Y$ and $N_\mu(x) = 0$ for $x \notin Y$. Let $m : K(X) \to E', m(A) = \mu(A)\phi$. Then $m \in M_t(X, E')$ and $N_m = \|\phi\|N_\mu$, which proves the Lemma.

Theorem 5.15 *If $f \in C_{b,k}(X, E)$, then f is (VR)-integrable with respect to every $m \in M_t(X, E')$. If E is polar, then the converse is also true.*

Proof: Assume that $f \in C_{b,k}(X, E)$ and let $m \in M_t(X, E')$. Let $\alpha > \|f|$ and $\epsilon > 0$. Then

$$D = \{x : \|f(x)\|N_m(x) \ge \epsilon\} \subset \{x : N_m(x) \ge \epsilon/\alpha\} = Z.$$

The set Z is τ_m-compact. Also, f is τ_m-continuous (by Theorem 5.13) and N_m is τ_m-upper semicontinuous. Thus D is a τ_m-closed subset of Z and hence D is τ_m-compact. Hence f is (VR)-integrable by Theorem 5.13.
Conversely, assume that E is polar and that the condition is satisfied. We show first that f is bounded. Assume the contrary. Since E is polar, there exists $\phi \in E'$ such that $\sup_{x \in X} |\phi(f(x))| = \infty$. Let $|\lambda| > 1$ and choose a sequence (a_n) of distinct elements of X such that $|\phi(a_n)| > |\lambda|^{2n}$ for all n. Define $m : K(X) \to E', m(A) = (\sum_{a_n \in A})\phi$. Then $m \in M_t(X, E')$. Let $a_n \in A \in K(X)$. If k is the smallest integer with $a_k \in A$, then, for $\phi(s) \neq 0$, we have

$$|m(A)s| = |\sum_{a_i \in A} \lambda^{-i}\phi(s)| = |\lambda^{-k}\phi(s)| \ge |\lambda^{-n}\phi(s)|,$$

and so $|m|(A) \ge |\lambda^{-n}|\|\phi\|$. On the other hand, suppose that $a_n \in A \in K(X)$. There exists a clopen neighborhood B of a_n contained in A and not containing any a_k for $k < n$. If now D is a clopen subset of B, then $|m(D)s| \le |\lambda^{-n}\phi(s)|$ and so $N_m(a_n) \le |m|(B) \le |\lambda^{-n}|\|\phi\|$. Thus $N_m(a_n) = |\lambda^{-n}|\|\phi\|$. But then

$$\|f\|_{N_m} \ge \sup_n \|f(a_n)\|\|\phi\||\lambda|^{-n} \ge \sup_n |\lambda|^{-n}|\phi(f(a_n)| = \infty,$$

a contradiction since f is (VR)-integrable. Thus f is bounded. Let next Y be a compact subset of X and let ϕ be a nonzero element of E'. By the preceding Lemma, there exists an $m \in M_t(X, E')$ such that $N_m(x) = \|\phi\|$ for $x \in Y$ and $N_m(x) = 0$ for $x \notin Y$. Given $\epsilon > 0$, there exists $g \in S(X, E)$ such that $\|f - g\|_{N_m} < \|\phi\|\epsilon$. Let $x_o \in Y$ and $V = \{x : \|g(x) - g(x_o)\| < \|\phi\|\epsilon\}$. If $x \in V \cap Y$, then

$$\|f(x) - f(x_o)\| \leq \max\{\|f(x) - g(x)\|, \|g(x) - g(x_o)\|, \|g(x_o) - f(x_o)\|\} \leq \epsilon,$$

which proves that $f|Y$ is continuous. This completes the proof.

Theorem 5.16 *Let $m \in M_t(X, E')$. If $f \in E^X$ is bounded and m-integrable, then $|\int f dm| \leq \|f\|_{N_m}$.*

Proof: Let $\epsilon > 0$. There exists a clopen partition $A_1, \ldots, A_n$ of X such that , for any clopen partition $D_1, \ldots, D_n$ of X which is a refinement of $A_1, \ldots, A_n$ and any $y_i \in D_i$, we have that $|\int f dm - \sum_{i=1}^{N} m(D_i) f(y_i)| < \epsilon$. Let $\epsilon_1 > 0$ be such that $\|f\|\epsilon_1 < \epsilon$. Choose $x_k \in A_k$ such that $\sup_{x \in A_k} N_m(x) < N_m(x_k) + \epsilon_1$. Now

$$|\int f dm - \sum_{k=1}^{n} m(A_k) f(x_k)| < \epsilon.$$

Moreover

$$|m(A_k) f(x_k)| \leq |m|(A_k)\|f(x_k)\| = [\sup_{y \in A_k} N_m(y)]\|f(x_k)\| \leq [\epsilon_1 + N_m(x_k)]\|f(x_k)\| \leq \epsilon + N_m(x_k)\|f(x_k)\|.$$

Thus

$$|\int f dm| \leq \max\{\epsilon, \max_k |m(A_k) f(x_k)|\} \leq \max\{\epsilon, \epsilon + \sup_{x \in X} N_m(x)\|f(x)\|\}.$$

Taking $\epsilon \to 0$, we get our result.

Theorem 5.17 *Let $m \in M_t(X, E')$ and $f \in E^X$ a bounded function. If f is both integrable and (VR)-integrable with respect to m, then $\int f dm = (VR) \int f dm$.*

Proof: There exists a sequence (g_n) in $S(X, E)$ such that $\|f - g_n\|_{N_m} \to 0$. Since $f - g_n$ is m-integrable and bounded, we have

$$|\int f dm - \int g_n dm| \leq \|f - g_n\|_{N_m} \to 0.$$

Thus,

$$\int f dm = \lim \int g_n dm = (VR) \int f dm.$$

Theorem 5.18 *Let $m \in M_t(X, E')$. For a bounded $f \in E^X$, the following are equivalent:*
(1)f is (VR)-integrable with respect to m.
(2) For every $\epsilon > 0$, $f|X_{m,\epsilon}$ is continuous.
(3) f is τ_m-continuous.
(4) f is (VR)-integrable with respect to $\bar{m}$.
In each of the above cases, we have

$$(VR) \int f dm = (VR) \int f d\bar{m} = \int f dm.$$

Proof: (2) is equivalent to (3) and (1) is equivalent to (4) by Proposition 5.8 and Corollary 5.12. Also (1) implies (2) by Theorem 5.13. Finally, assume that (2) holds and let $d > \|f\|$. Then

$$D = \{x : \|f(x)\|N_m(x) \geq \epsilon\} \subset \{x : N_m(x) \geq \epsilon/d\} = Z.$$

Since f is τ_m-continuous and N_m τ_m-upper semicontinuous, it follows that D is a τ_m-closed subset of the τ_m-compact Z and hence it is τ_m-compact. By Threorem 5.13, f is (VR)-integrable with respect to m. In each of the above cases f is τ_m-continuous and so it is m-integrable and thus

$$(VR)\int f dm = (VR)\int f d\bar{m} = \int f dm$$

by Corollary 5.12 and Theorem 5.17. This completes the proof.

6 Q-Integrals

Theorem 6.1 *Let $m \in M(X, E')$ and $f \in E^X$. Then f is m-integrable iff the following condition is satisfied: For each $\epsilon > 0$, there exists a clopen partition $\{A_1, \ldots, A_n\}$ of X such that, for every x, y which are in the same A_k and any clopen subset B of A_k we have $|m(B)(f(x) - f(y))| \leq \epsilon$.*

Proof: Assume that f is m-integrable and let $\epsilon > 0$. There exists a clopen partition $\{A_1, \ldots, A_n\}$ of X such that, for every clopen partition $\{D_1, \ldots, D_N\}$ of X which is a refinement of $\{A_1, \ldots, A_n\}$ and any choice of $x_k \in D_k$ we have that $|\int f dm - \sum_{k=1}^{N} m(D_k)f(x_k)| \leq \epsilon$. Let now x, y be in some A_i and let B be a clopen subset of A_i. We will show that $|m(B)(f(x) - f(y))| \leq \epsilon$. To prove this, we consider the three possible cases:
Case I. $x, y \in B$. Then it is clear that $|m(B)(f(x) - f(y))| \leq \epsilon$.
Case II. $x, y \in D = A_i \setminus B$. Assume, by way of contradiction, that $|m(B)(f(x) - f(y))| > \epsilon$. Since $\epsilon \geq |m(A_i)(f(x) - f(y))| = |m(B)(f(x) - f(y)) + m(D)(f(x) - f(y))|$, we would have that $|m(B)(f(x) - f(y))| = |m(D)(f(x) - f(y))| \leq \epsilon$, a contradiction.
Case III. $x \in B$ and $y \in D$ (say). Then $|m(A_i)f(y) - [m(B)f(x) + m(D)f(y)]| \leq \epsilon$, i.e. $|m(B)(f(x) - f(y))| \leq \epsilon$.
Thus the condition is satisfied. Conversely, suppose that the condition holds and let $\epsilon > 0$. Let $\{A_1, \ldots, A_n\}$ be as in the condition and let $x_k \in A_k$. If $\{B_1, \ldots, B_N\}$ is a clopen partition of X which is a refinement of $\{A_1, \ldots, A_n\}$ and if $y_j \in B_j$, then for $B_j \subset A_k$, we have that $|m(B_j)[f(y_j) - f(x_k)]| \leq \epsilon$, and thus $|\sum_{k=1}^{n} m(A_k)f(x_k) - \sum_{j=1}^{N} m(B_j)f(y_j)| \leq \epsilon$. This clearly proves that f is m-integrable and hence the result follows.

Let now $m \in M_\tau(X, E')$ and $f \in E^X$. We define $Q_{m,f}$ on X by

$$Q_{m,f}(x) = \inf_{x \in A \in K(X)} \sup\{|m(B)f(x)| : B \subset A, B \in K(X)\}.$$

Also, for $A \in K(X)$, we define

$$\|f\|_{A,Q_m} = \sup_{x \in A} Q_{m,f}(x), \quad \|f\|_{Q_m} = \|f\|_{X,Q_m}.$$

Lemma 6.2 *If* $g = \sum_{k=1}^n \chi_{A_k} s_k$, *where* $A_k \in K(X), s_k \in E$, *then* $|\sum_{k=1}^n m(A_k)s_k| \leq \|g\|_{Q_m}$.

Proof: We may assume that the A_k are pairwise disjoint. We prove first that, for $A \in K(X), s \in E, h = \chi_A s$, we have that $|m(A)s| \leq \sup_{x \in A} Q_{m,h}(x)$. Indeed, let $\theta > \sup_{x \in A} Q_{m,h}(x)$. For each $x \in A$, there exists a clopen neighborhood V_x of x contained in A such that $|m(B)h(x)| = |m(B)s| < \theta$ for evey clopen set B contained in V_x. Let $\mu = ms$ be defined by $\mu(B) = m(B)s, B \in K(X)$. Then $\mu \in M_\tau(X)$. Since $|\mu|(V_x) < \theta$ for every $x \in A$, it follows that $|\mu|(A) \leq \theta$. Thus $|m(A)s| \leq \theta$, which proves that $|m(A)s| \leq sup_{x \in A} Q_{m,h}(x)$. If $h_k = \chi_{A_k} s_k$, then for $x \in A_k$ we have $Q_{m,h_k}(x) = Q_{m,g}(x)$ and so $|m(A_k)s_k| \leq sup_{x \in A_k} Q_{m,g}(x)$ which clearly completes the proof.

As we have shown in the proof of Theorem 6.1, we have the following

Theorem 6.3 *Let* $m \in M_\tau(X, E')$ *and let* $f \in E^X$ *be* m*-integrable. Then, given* $\epsilon > 0$, *there exists a clopen partition* $\{A_1, \ldots, A_n\}$ *of* X *such that for any* $x_k \in A_k$ *and* $g = \sum_{k=1}^n \chi_{A_k} f(x_k)$ *we have that* $|\int f dm - \sum_{k=1}^n m(A_k) f(x_k)| \leq \epsilon$ *and* $\|f - g\|_{Q_m} \leq \epsilon$.

Lemma 6.4 *Let* $m \in M_\tau(X, E')$ *and let* $p \in cs(E)$ *be such that* $m_p(X) < \infty$. *If* $f \in E^X$ *is bounded, then* $\|f\|_{Q_m} \leq \|f\|_p m_p(X)$.

Proof: It folows from the fact that, for $B \in K(X)$, we have $|m(B)f(x)| \leq m_p(X)p(f(x))$.

Lemma 6.5 *Let* $m \in M_\tau(X, E')$ *and let* $f \in E^X$ *be* m*-integrable. Then* $\|f\|_{Q_m} < \infty$.

Proof: There exists $g \in S(X)$ such that $\|f - g\|_{Q_m} \leq 1$. Let $p \in cs(E)$ be such that $m_p(X) \leq 1$. Then

$$\|f\|_{Q_m} \leq \max\{1, \|g\|_{Q_m}\} \leq \max\{1, m_p(X)\|g\|_p\}.$$

Lemma 6.6 *Let* $m \in M_\tau(X, E')$. *If* $f \in E^X$ *is* m*-integrable, then* $|\int f dm| \leq \|f\|_{Q_m}$.

Proof: Given $\epsilon > 0$, let $\{A_1, \ldots, A_n\}$ be a clopen plartition of X such that, for every clopen partition $\{D_1, \ldots, D_N\}$ of X which is a refinement of $\{A_1, \ldots, A_n\}$ and any choice of $x_k \in D_k$ we have that $|\int f dm - \sum_{k=1}^N m(D_k) f(x_k)| \leq \epsilon$. Let $x_k \in A_k$ and $g = \sum_{k=1}^n \chi_{A_k} f(x_k)$. Let $x \in A_k$. There exist a clopen subset D of A_k with $x \in D$ such that $|m(B)f(x)| < Q_{m,f}(x) + \epsilon$ for every clopen set $B \subset D$. Thus, for $B \subset D$, we have

$$|m(B)g(x)| = |m(B)f(x_k)| \leq \max\{|m(B)(f(x_k) - f(x))|, |m(B)f(x)|\} \leq Q_{m,f}(x) + \epsilon$$

and so $Q_{m,g}(x) \leq Q_{m,f}(x) + \epsilon$. Now

$$|\int f dm| \leq \max\{\epsilon, |\sum_{k=1}^{n} m(A_k) f(x_k)|\} \leq \max\{\epsilon, \sup_x Q_{m,g}(x)\} \leq \sup_{x\in X} Q_{m,p}(x) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the result follows.

Lemma 6.7 *Let $m \in M_\tau(X, E')$ and $f \in E^X$. If $(g_n) \subset S(X)$ is such that $\|f - g_n\|_{Q_m} \to 0$, then the $\lim_{n\to\infty} \int g_n dm$ exists. Moreover, if (h_n) is another sequence in $S(X)$ such that $\|f - h_n\|_{Q_m} \to 0$, then $\lim_{n\to\infty} \int g_n dm = \lim_{n\to\infty} \int h_n dm$.*

Proof: Since $|\int g_n dm - \int g_k dm| \leq \|g_n - g_k\|_{Q_m} \leq \max\{\|g_n - f\|_{Q_m}, \|f - g_k\|_{Q_m}\}$, it follows that the $\lim_{n\to\infty} \int g_n dm$ exists. If (h_n) is another sequence in $S(X)$ such that $\|f - h_n\|_{Q_m} \to 0$, then

$$|\int g_n dm - \int h_n dm| \leq \max\{\|g_n - f\|_{Q_m}, \|f - h_n\|_{Q_m}\} \to 0.$$

Thus the result follows.

Definition 6.8 *Let $m \in M_\tau(X, E')$. A function $f \in E^X$ is said to be Q-integrable with respect to m if there exists a sequence (g_n) in $S(X)$ such that $\|f - g_n\|_{Q_m} \to 0$. In this case, the $\lim_{n\to\infty} \int g_n dm$ is called the Q-integral of f and will be denoted by $(Q) \int f dm$.*

By what we have shown above, if $f \in E^X$ is m-integrable for some $m \in M_\tau(X, E')$, then f is Q-integrable and $\int f dm = (Q) \int f dm$.

Theorem 6.9 *If $m \in M_t(X, E')$, then every $f \in E^X$ which is (VR)-integrable with respect to m, is also Q-integrable and $(VR) \int f dm = (Q) \int f dm$.*

Proof: It follows from the fact that, if $m \in M_{t,p}(X, E')$, then for each $h \in E^X$ we have $Q_{m,h}(x) \leq N_{m,p}(x) p(h(x))$ for every $x \in X$.

Theorem 6.10 *Assume that E is polar and let $f \in E^X$. If f is Q-integrable with respect to m for each $m \in M_\tau(X, E')$, then f is bounded.*

Proof: Assume that f is not bounded. Since E is polar, there exists $\phi \in E'$ with $\sup_{x\in X} |\phi(f(x))| = \infty$. Let $|\lambda| > 1$ and choose a sequence (a_n) of distinct elements of X such that $|\phi(f(a_n)) > |\lambda|^{2n}$ for all n. Let $m : K(X) \to E', m(A) = (\sum_{a_n \in A} \lambda^{-n})\phi$. Then $m \in M_\tau(X, E')$. Let now $a_n \in A \in K(X)$ and let D be a clopen subset of A containing a_n and not containing any a_k for $k < n$. Then

$$|m(D) f(a_n)| = |(\sum_{a_k \in D} \lambda^{-k})\phi(f(a_n))| = |\lambda|^{-n} |\phi(f(a_n))| \geq |\lambda|^n.$$

This proves that $Q_{m,f}(a_n) \geq |\lambda|^n$ and thus $\|f\|_{Q_m} = \infty$, which implies that f is not Q-integrable with respect to m (in view of Lemma 6.5). This contradiction completes the proof.

For an $m \in M_\tau(X, E')$, define q_m on $C_b(X, E)$ by $q_m(f) = \|f\|_{Q_m}$.

Theorem 6.11 *If $m \in M_\tau(X, E')$, then q_m is β-continuous.*

Proof: It is easy to see that q_m is a non-Archimedean seminorm on $C_b(X, E)$. To prove that q_m is β_o-continuous, let $G \in \Omega$.There exists a decreasing net (A_δ) of clopen subsets of X such that $G = \bigcap \bar{A_\delta}^{\beta_o X}$. Let $p \in cs(E)$ be such that $m_p(X) < \infty$ and $m_p(A_\delta) \to 0$. Let $r > 0$ and choose δ such that $m_p(A_\delta) < 1/r$. The closure in $\beta_o X$ of the set $X \setminus A_\delta$ is disjoint from G. Now

$$V = \{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1/m_p(X)\} \subset \{f \in C_b(X, E) : q_m(f) \leq 1\}.$$

Indeed, let $f \in V$. If $x \in A_\delta$, then $Q_{m,f}(x) \leq m_p(A_\delta)p(f(x)) \leq 1$. Also, for $x \in B$ and $D \subset B$, we have $|m(D)f(x)| \leq m_p(X)p(f(x)) \leq 1$ and thus $\|f\|_{Q_m} \leq 1$. This proves that the set $W = \{f \in C_b(X, E) : q_m(f) \leq 1\}$ is a β_G-neighborhood of zero for each $G \in \Omega$ and hence it is a β-neighborhood. Thus q_m is β- continuous.

References

[1] J. Aguayo, N de Grande-de Kimpe and S. Navarro, *Strict locally convex topologies on $BC(X, \mathbb{K})$*, in: P-adic Functional Analysis, edided by W. H. Schikhof, C. Perez- Garcia and J. Kakol, Lecture Notes in Pure and Applied Mathematics, vol. 192, Marcel Dekker, New York (1997), 1-9.

[2] J. Aguayo, N de Grande-de Kimpe and S. Navarro, *Zero-dimensional pseudo-compact and ultraparacompact spaces* , in: P-adic Functional Analysis, edided by W. H. Schikhof, C. Perez- Garcia and J. Kakol, Lecture Notes in Pure and Applied Mathematics, vol. 192, Marcel Dekker, New York (1997), 11-17.

[3] J. Aguayo, N de Grande-de Kimpe and S. Navarro, *Strict topologies and duals in spaces of functions* , in: P-adic Functional Analysis, edided by J. Kakol, N. de Grande-de Kimpe and C. Perez- Garcia , Lecture Notes in Pure and Applied Mathematics, vol. 207, Marcel Dekker, New York (1999), 1-10.

[4] G. Bachman, E. Beckenstein, L. Narici and S. Warner, *Rings of continuous functions with values in a topological field*, Trans. Amer. Math. Soc. **204** (1975), 91-112.

[5] N. de Grande-de Kimpe and S. Navarro, *Non-Archimedean nuclearity and spaces of continuous functions* , Indag. Math., N.S. **2** (2) (1991), 201-206.

[6] A. K. Katsaras, *Duals of non-Archimedean vector-valued function spaces*, Bull. Greek Math. Soc. **22** (1981), 25-43.

[7] A. K. Katsaras, *The strict topology in non-Archimedean vector-valued function spaces*, Proc. Kon. Ned. Akad. Wet. A **87**(2) (1984), 189-201

[8] A. K. Katsaras, *Strict topologies in non-Archimedean function spaces*, Intern. J. Math. and Math. Sci. **7**(1) (1984), 23-33.

[9] A. K. Katsaras, *On the strict topology in non-Archimedean spaces of continuous functions*, Glasnik Mat. Vol. **35** (55) (2000), 283-305.

[10] A. K. Katsaras, *Separable measures and strict topologies on spaces of non-Archimedean valued functions*, Technical Report, Vol **5** (2001), University of Ioannina, Greece.

[11] A. K. Katsaras, *Strict topologies and vector-measures on non-Archimedean spaces* (Preprint)

[12] A. K. Katsaras and A. Beloyiannis, *Tensor products of non-Archimedean weighted spaces of continuous functions*, Georgian J. Math. Vol. **6**, No 1(1999), 33-44.

[13] C. Perez-Garcia, *P-adic Ascoli theorems and compactoid polynomials*, Indag. Math., N. S. **3**(2) (1993), 203-210.

[14] J. B. Prolla, Approximation of vector-valued functions, North Holland Publ. Co., Amsterdam, New York, Oxford, 1977.

[15] W. H. Schikhof, *Locally convex spaces over non-spherically complete fields I, II*, Bull. Soc. Math. Belg., Ser. B, **38** (1986), 187-224

[16] A. C. M. van Rooij, Non-Archimedean Functional Analysis, New York and Basel, Marcel Dekker, 1978.